ON COMPUTING EIGENSOLUTION SENSITIVITY DATA USING FREE VIBRATION SOLUTIONS

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SUMMARY

A simplified method of computing eigensolution sensitivity derivatives in structural dynamics is developed in this paper. It is shown that if the elements of stiffness and mass matrices associated with a design variable are homogeneous functions of that design variable, then eigenvalue derivatives can be computed from element strain and kinetic energies. Furthermore, if cross-mode energies are known, eigensolution derivatives of modified systems can be computed approximately using assume mode reanalysis formulation. A ten bar truss example is used to illustrate the present formulations.

INTRODUCTION

The usefulness of eigensolution sensitivity derivatives in structural dynamics research is well known. The sensitivity data can be used for approximate reanalysis, analytical model improvement, assessment of design trend as well as structural optimization with eigenvalue constraints. When applied to larger discrete structural models, these applications typically require long and expensive computer runs and usually the predominate contributor to the computing time was the calculation of derivatives. Thus efficient eigensolution sensitivity analysis procedures would be very useful in structural dynamic research. It is the purpose of this paper to develop, under certain conditions, efficient eigensolution analysis procedures using free vibration data.

The equations for computing derivatives of eigenvalues and eigenvectors for free vibration of undamped structures were known for a long time. Only recently have these methods been implemented in some general-purpose finite-element programs. In this paper, a simple method is developed which can be used to compute the eigenvalue derivatives for a large class of problems by exploiting the similarity between the equations for eigenvalue derivatives and element strain and kinetic energies. Furthermore, if the cross-mode element energy data are available, the approximate eigenvector derivatives can also be computed using a truncated modal expansion expression. The approximate second derivatives of eigenvalues can then be computed. Additionally, with the cross-mode strain energy data, the eigenvalue derivatives of a modified structure can be computed using assumed mode reanalysis formulation. Numerical examples will be presented to illustrate the various formulations.

EIGENSOLUTION SENSITIVITY IN STRUCTURAL DYNAMICS

The general problem is to compute the rate of change (or derivatives) of eignevalues and eigenvectors with respect to design variables for the following generalized eigenvalue problem in structural dynamics.

$$K \phi = \lambda M \phi \tag{1}$$

Much research has addressed this problem in the past two decades. A comprehensive survey of literature can be found in a recent paper by Adelman and Haftka [1]. The equations for first order eigenvalue and eigenvector derivatives as well as second order eigenvalue derivatives are summarized below:

Eigenvalue Derivative:

$$\frac{\partial \lambda_\ell}{\partial x_r} = \phi_\ell^T \frac{\partial K}{\partial x_r} \phi_\ell - \lambda_\ell \phi_\ell^T \frac{\partial M}{\partial x_r} \phi_\ell \tag{2}$$

Eigenvector Derivative:

$$\frac{\partial \phi_\ell}{\partial x_r} = \sum_{j=1}^{n} A_{\ell ij} \phi_j \tag{3}$$

where for $\ell \neq j$

$$A_{\ell rj} = \phi_j^T \frac{\partial Z_\ell}{\partial x_r} \phi_\ell / (\lambda_\ell - \lambda_j) \tag{4}$$

$$Z_\ell = K - \lambda_\ell M \tag{5}$$

and

$$A_{\ell r \ell} = -\frac{1}{2} \phi_\ell^T \frac{\partial M}{\partial x_r} \phi_\ell \tag{6}$$

Second Derivative of Eigenvalues:

$$\frac{\partial^2 \lambda_\ell}{\partial x_r \partial x_s} = Y_\ell + \phi_\ell^T \left(\frac{\partial Z_\ell}{\partial x_r} \frac{\partial \phi_\ell}{\partial x_s} + \frac{\partial Z_\ell}{\partial x_s} \frac{\partial \phi_\ell}{\partial x_r} \right) \tag{7}$$

where

$$Y_\ell = \phi_\ell^T \left(\frac{\partial^2 K}{\partial x_r \partial x_s} - \lambda_\ell \frac{\partial^2 M}{\partial x_r \partial x_s} - \frac{\partial \lambda_\ell}{\partial x_s} \frac{\partial M}{\partial x_r} - \frac{\partial \lambda_\ell}{\partial x_r} \frac{\partial M}{\partial x_s} \right) \phi_\ell \tag{8}$$

Note that in the above equations, the mode shapes are normalized to unit generalized mass, i.e.

$$\phi_\ell^T M \phi_\ell = 1.0$$

For the eigenvector derivatives, if less than full modes are used, Eq. (2) is an approximate expression. These will lead to approximate second order derivatives of eigenvalues.

The above equations have been developed in the literature for some time. For example, Equations (2) and (3) can be found in Fox and Kapoor [2] and Eq. (7) was reported by Miura and Schmit [3]. It should be noted in passing that there are some recently developed algebraic methods [4-5] which can be used to compute eigenvector derivatives without using modal expansions.

The difficulty of applying the aforementioned equations appears to be the calculation of derivatives of stiffness and mass matrices with respect to design variables. In the next section it will be shown that under certain assumptions, we can circumvent the calculation of $\partial K/\partial x_i$ and $\partial M/\partial x_i$ in implementing these above equations.

SIMPLIFYING ASSUMPTIONS

In general, the system stiffness and mass matrices in Eq. (1) can be written as

$$K = K_c + \sum_{i=1}^{ND} K_i \tag{9}$$

$$M = M_c + \sum_{i=1}^{ND} M_i \tag{10}$$

where

K_c = contribution to stiffness matrix due to structural elements that are to remain constant during the design process.

M_c = contribution to mass matrix due to the masses of the unchanged elements as well as nonstructural masses.

K_i, M_i = contributions to stiffness and mass matrices respectively due to elements controlled by design variable x_i.

To develop simplified efficient methods for eigensolution sensitivity analysis, the elements of the matrices K_i and M_i are assumed to be homogeneous functions of design variables. That is the matrices K_i and M_i have the form

$$K_i = (x_i)^{\beta_i} K_i^* \tag{11}$$

$$M_i = (x_i)^{\gamma_i} M_i^* \tag{12}$$

where K_i^* and M_i^* are constant matrices. Furthermore, define non-dimensional design parameters

$$\alpha_i = \left(\frac{x_i}{x_{i0}}\right)^{p_i} \tag{13}$$

$$\bar{\alpha}_i = \left(\frac{x_i}{x_{i0}}\right)^{\gamma_i} \tag{14}$$

Then

$$K_i = (x_i)^{\beta_i} K_i^* = \left(\frac{x_i}{x_{i0}}\right)^{\beta_i} (x_{i0})^{\beta_i} K_i^*$$

or

$$K_i = \alpha_i K_{i0} \tag{15}$$

Similarly,

$$M_i = \bar{\alpha}_i M_{i0} \tag{16}$$

where K_{i0} and M_{i0} are stiffness and mass matrices due to design variable x_i at its nominal value x_{i0}.

Based on the above assumptions, the derivatives of stiffness and mass matrices with respective to design variables can be computed readily:

$$\frac{\partial K}{\partial x_i} = \frac{\partial K_i}{\partial x_i} = \frac{\partial \alpha_i}{\partial x_i} K_{i0}$$

or

$$\frac{\partial K_i}{\partial x_i} = \frac{\beta_i}{x_i} K_{i0} \tag{17}$$

Similarly, we can derive

$$\frac{\partial M}{\partial x_i} = \frac{\gamma_i}{x_i} M_{i0} \tag{18}$$

It should be noted that at the nominal design, $\alpha_i = \bar{\alpha}_i = 1$. With these simplifications, the eigenvalue derivatives can be computed readily.

RATE OF CHANGE OF EIGENVALUES

Using (17) and (18) with $\alpha = \bar{\alpha}_i = 1$, the eigenvalue derivative, eq. (2) becomes

$$\frac{\partial \lambda_\ell}{\partial x_r} = \frac{\beta_r}{x_r} \phi_\ell^T K_{r0} \phi_\ell - \frac{\gamma_r}{x_r} \lambda_\ell \phi_\ell^T M_{r0} \phi_\ell \tag{19}$$

Define

$$V_{\ell r \ell} = \frac{1}{2} \phi_\ell^T K_{r0} \phi_\ell \tag{20}$$

$$T_{\ell r \ell} = \frac{1}{2} \lambda_\ell \phi_\ell^T M_{r0} \phi_\ell \tag{21}$$

Then Eq. (19) can be written as

$$\frac{\partial \lambda_\ell}{\partial x_r} = 2\left(\frac{\beta_r}{x_r} V_{\ell r \ell} - \frac{\gamma_r}{x_r} T_{\ell r \ell}\right) \tag{22}$$

Note that from Eqs. (20) and (21), $V_{\ell r \ell}$ and $T_{\ell r \ell}$ can be interpreted as the strain and kinetic energy respectively of elements associated with design variable x_r.

Thus, given β_r and γ_r, the rate of change of eigenvalues can be computed from the energies associated with this design variable. Since most general-purpose finite-element codes provide element strain energy as an output option, one way to implement (22) is to calculate $V_{\ell r \ell}$ and $T_{\ell r \ell}$ by summing strain energy and kinetic energy for all elements controlled by design variable x_r.

In the above formulation, we have made use of the form of the stiffness and mass matrices, Eqs. (11), (12). Not all structural elements can fit into these models but some important cases do. Some of these are tabulated in Table 1.

Using Eqs. (15) to (18), it is possible to derive explicit equations for eigenvector derivatives as well as second-order derivatives of eigenvalues in terms of energies associated with various design variables. These are quite tedious and have not been accomplished so far. In the following, we will discuss the special case of $\beta_i = \gamma_i$.

EIGENSOLUTION SENSITIVITIES FOR THE CASE $\beta_i = \gamma_i$

For this special case, we can use chain rules to rewrite sensitivity derivatives as:

$$\frac{\partial \lambda_\ell}{\partial x_r} = \frac{\partial \lambda_\ell}{\partial \alpha_r} \frac{\partial \alpha_r}{\partial x_r} \tag{23}$$

$$\frac{\partial \phi_\ell}{\partial x_r} = \frac{\partial \phi_\ell}{\partial \alpha_r} \frac{\partial \alpha_r}{\partial x_r} \tag{24}$$

$$\frac{\partial^2 \lambda_\ell}{\partial x_r \partial x_s} = \frac{\partial^2 \lambda_\ell}{\partial x_r \partial x_s} \frac{\partial \alpha_r}{\partial x_r} \frac{\partial \alpha_s}{\partial x_s} \tag{25}$$

Thus, it remains to find $\partial \lambda_\ell / \partial \alpha_r$, $\partial \phi \partial \lambda_\ell / \partial \alpha_r$ and $\partial^2 \lambda_\ell / \partial \alpha_r \partial \alpha_s$. Note that

$$\frac{\partial K_r}{\partial \alpha_r} = K_{r0} \tag{26}$$

$$\frac{\partial M_r}{\partial \alpha_r} = M_{r0} \tag{27}$$

Using Eqs. (26), (27) and replacing all x_r, x_s in Eqs. (2) to (8) by α_r and α_s, and making use of the orthogonality properties of normal modes

$$\phi_\ell^T M \phi_j = 0 \qquad \text{if } \ell \neq j$$

as well as the linearity assumptions (Eqs. (15) and (16)), we can derive, after considerable algebraic manipulation, the following results:

$$\frac{\partial \lambda_\ell}{\partial \alpha_r} = C_{\ell r \ell} \tag{28}$$

$$\frac{\partial \phi_\ell}{\partial \alpha_r} = \sum_{j=1}^{n} \bar{C}_{\ell r j} \phi_j \tag{29}$$

$$\frac{\partial^2 \lambda_\ell}{\partial \alpha_r \partial \alpha_s} = 2\left[\left(\frac{\partial \lambda_\ell}{\partial \alpha_s} C_{\ell r \ell} + \frac{\partial \lambda_\ell}{\partial \alpha_r} C_{\ell s \ell}\right) + \sum_{\substack{j=1 \\ j \neq \ell}}^{n} \bar{C}_{\ell r j} C_{\ell s j}\right] \tag{30}$$

where

$$C_{\ell r j} = 2(V_{\ell r j} - T_{\ell r j}) \tag{31}$$

$$V_{\ell r j} = \frac{1}{2} \phi_\ell^T K_{ro} \phi_j \tag{32}$$

$$T_{\ell r j} = \frac{1}{2} \lambda_\ell \phi_\ell^T M_{ro} \phi_j \tag{33}$$

$$\bar{C}_{\ell r j} = \frac{C_{\ell r j}}{\lambda_\ell - \lambda_j} \qquad \ell \neq j \tag{34}$$

$$\bar{C}_{\ell r \ell} = -\frac{T_{\ell r \ell}}{\lambda_\ell} \tag{35}$$

It should be noted that $V_{\ell r j}$ can be considered as the "cross mode" strain energy, since it is the work done by the elastic force in jth mode (i.e. $K_r \phi_j$) moving through displacement in the ℓth mode. Similarly, $T_{\ell r j}$ can be considered as the "cross mode" kinetic energy. Thus, eigensolution sensitivity derivatives can be computed readily when these energy terms become available.

SENSITIVITY DERIVATIVES FOR MODIFIED SYSTEMS

In iterative analysis, we frequently require the eigensolution derivatives of a system different from the nominal design. In these situations, assumed mode reanalysis [6-7] appears to be very efficient. Let ΔK and ΔM denote the change to stiffness and mass matrices, respectively. Then, in term of α_i, we have

$$\Delta K = \sum_{i=1}^{ND} (\alpha_i - 1) K_{io} \tag{36}$$

and

$$\Delta M = \sum_{i=1}^{ND} (\alpha_i - 1) M_{io} \tag{37}$$

Following the development in Ref. 7, the eigensolution of the modified system can be computed approximately by solving the following reduced eigenvalue problem

$$\bar{K}\, q = \lambda\, \bar{M}\, q \tag{38}$$

where

$$\bar{K} = \Phi^T (K + \Delta K)\Phi$$

$$= [\lambda_0] + \Sigma\, (\alpha_i - 1)\, \tilde{K}_i \tag{39}$$

$$\bar{M} = [I] + \Sigma\ \ (\alpha_i - 1)\, \tilde{M}_i \tag{40}$$

where Φ is the truncated modal matrix of the original system, and

$$\tilde{K}_i = \Phi^T\, K_{i0}\, \Phi \tag{41}$$

$$\tilde{M}_i = \Phi^T\, M_{i0}\, \Phi \tag{42}$$

Once (37) is solved, the eigenvectors of the modified system ψ_i, in terms of physical coordinates, can be completed from

$$\psi_i = \Phi\, q_i \tag{43}$$

For modified systems, the eigenvalue derivatives, Eqs. (28) and (30) are still applicable except $V_{\ell rj}$ and $T_{\ell rj}$ are now defined by

$$V_{\ell rj} = \frac{1}{2}\, q_\ell^T\, \tilde{K}_r\, q_\ell \tag{44}$$

$$T_{\ell rj} = \frac{1}{2}\, \lambda_\ell\, q_\ell^T\, \tilde{M}_r\, q_\ell \tag{45}$$

and the eigenvector derivatives can be computed from

$$\frac{\partial \psi_\ell}{\partial \alpha_r} = \Phi\, \frac{\partial q_\ell}{\partial \alpha_r} \tag{46}$$

where

$$\frac{\partial q_\ell}{\partial \alpha_r} = \sum_{j=1}^{n} \bar{C}_{\ell rj}\, q_j \tag{47}$$

$\bar{C}_{\ell rj}$ is as defined by (34) or (35) with $V_{\ell rj}$, $T_{\ell rj}$ defined by (44) and (45).

DISCUSSION

In Eq. (7), the second-order derivatives of eigenvalues are shown to be dependent on eigenvector derivatives. In the present formulation, we can compute $\partial^2\lambda_\ell/\partial\alpha_r\partial\alpha_s$ Using Eq. (34) without the need to compute eigenvector derivatives explicitly. Once the derivatives with respective to α's are known, chain rules can be used to compute the derivatives with respective to design variables x's (Eqs. (23) to (25)).

NUMERICAL EXAMPLE

The assumed mode reanalysis sensitivity derivative formulation has been implemented in a program which post-processes MSC/NASTRAN generated data. The first-order sensitivity data have been applied to improve analytical model using measured modal data [8] as well as synthesis of structures with multiple frequency constraints [9]. Recently the second order derivatives of eigenvalues (Eq. (30)) has also been implemented.

A ten-bar cantilever truss structure, Fig. 1, is used to test the program. The ten members are grouped into 4 design variables as indicated in Figure 1. Starting with a uniform structure with cross sectional area 10 in^2 for all design variables, the optimal design program described in Ref. 7 is used to mode the first two natural frequencies from 13.3 and 37.8 Hz to 16 and 39.3 Hz, respectively. This is accomplished by a sequential linear programming formulation [7,9]. At each intermediate design, the eigenvalue derivatives are computed using reanalysis formulations. Table 2 defines the design history. Specifically, the designs at iteration No. A-0 and B-0 are analyzed exactly using MSC/NASTRAN. Three iterations are shown after each exact analysis. The eigensolution at designs A-1 to A-3 and B-1 to B-3 are computed using assumed mode reanalysis formulations. Four modes are used in each case. The first two natural frequencies are tabulated in Table 3. Also shown in Table 3 are the corresponding exact frequencies. From Table 3, it can be seen that the accuracy in frequency of assumed mode reanalysis formulation is very good. Tables 4 to 7 summarize $\partial\lambda_1/\partial_1$, $\partial\lambda_1/\partial x_2$, $\partial\lambda_2/\partial x_1$ and $\partial\lambda_2/\partial x_2$, respectively. The results of these tables indicate that the sensitivity derivatives of modified system can be predicted quite accurately using the assumed-mode reanalysis formulation.

CONCLUDING REMARKS

General procedures for computing eigensolution sensitivity derivatives for a class of problems have been proposed in this paper. Detailed formulations have been carried out for a special case. It is shown that the eigenvalue derivative with a design variable can be computed from strain energy and kinetic energy for that design variable. Furthermore, when the cross mode energy terms are available, assumed mode method can be used for eigensolution as well as associated sensitivity reanalysis. This efficient formulation has proved to be very effective in synthesis of structures with multiple frequency constraints [7,9]. Additionally, the present approach can be implemented in a post-processor of any finite-element programs without the need to modify the source code.

Since the current formulation provides an efficient approach for computing second-order eigenvalue derivatives, it would appear that a second-order method for structural optimization with frequency constraints could be implemented efficiently. Finally, in view of the success of the formulation for the special case of $\beta_i = \gamma_i$, further development for the general case of $\beta_i \neq \gamma_i$ seems to be warranted.

SYMBOLS

K = system stiffness matrix
M = system mass matrix
ϕ_ℓ = eigenvector of the ℓth mode
λ_ℓ = eigenvalue of the ℓth mode
Φ = modal matrix of original system
$V_{\ell rj}$ = cross-mode strain energy
$T_{\ell rj}$ = cross-mode kinetic energy
x_r = rth design variable
ΔK = modification in stiffness matrix
ΔM = modification in mass matrix
ψ_ℓ = eigenvector of the th mode of the modified systems
N = number of dof of the system
n = number of modes computed, $n < N$
ND = number of design variables

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TABLE I. - STIFFNESS AND MASS EXPONENTS FOR SEVERAL COMMON STRUCTURAL ELEMENTS

Element	Design variable	β	γ
Truss	Cross	1	1
Membrane	Thickness	1	1
Plate bending	Thickness	3	1
Beam bending	Cross-sectional area*	2	1
Beam bending	Section area moment of inertia*	1	0.5

*Circular cross section

TABLE II. - DESIGN HISTORY OF TEN BAR TRUSS

Iteration No.	x_1	2	x_3	x_4
A-0	10.0	10.0	10.0	10.0
A-1	12.76	8.88	5.0	5.0
A-2	10.39	8.15	5.0	5.0
A-3	10.44	8.29	5.0	5.0
B-0	10.44	8.29	5.0	5.0
B-1	7.92	7.80	3.44	3.44
B-2	7.47	7.18	2.90	2.90
B-3	7.19	7.0	2.68	2.68

TABLE III. - COMPARISONS OF NATURAL FREQUENCIES

Case No.	f_1(Hz) Approximate	f_1(Hz) Exact	Error (%)	f_2(Hz) Approximate	f_2(Hz) Exact	Error (%)
A-1	16.88	16.60	1.71	40.16	39.03	2.88
A-2	15.96	15.78	1.14	39.85	38.99	2.19
A-3	15.99	15.82	1.10	40.00	39.11	1.27
B-1	15.90	15.89	0.11	40.23	39.87	0.88
B-2	15.99	15.98	0.03	39.98	39.44	1.37
B-3	15.99	15.99	0.00	40.00	39.33	1.71

TABLE IV. - COMPARISON OF $\partial\lambda_1/\partial x_1$

Case No.	$\partial\lambda/\mu x$ Eq. (28)	$\partial\lambda/\mu x$ Exact	Error (%)
A-1	418.9	454.0	7.7
A-2	494.2	509.2	2.9
A-3	495.8	505.9	1.9
B-1	557.2	578.3	3.7
B-2	607.3	609.2	0.3
B-3	635.8	620.0	2.6

TABLE V. - COMPARISON OF $\partial\lambda_1/\partial x_2$

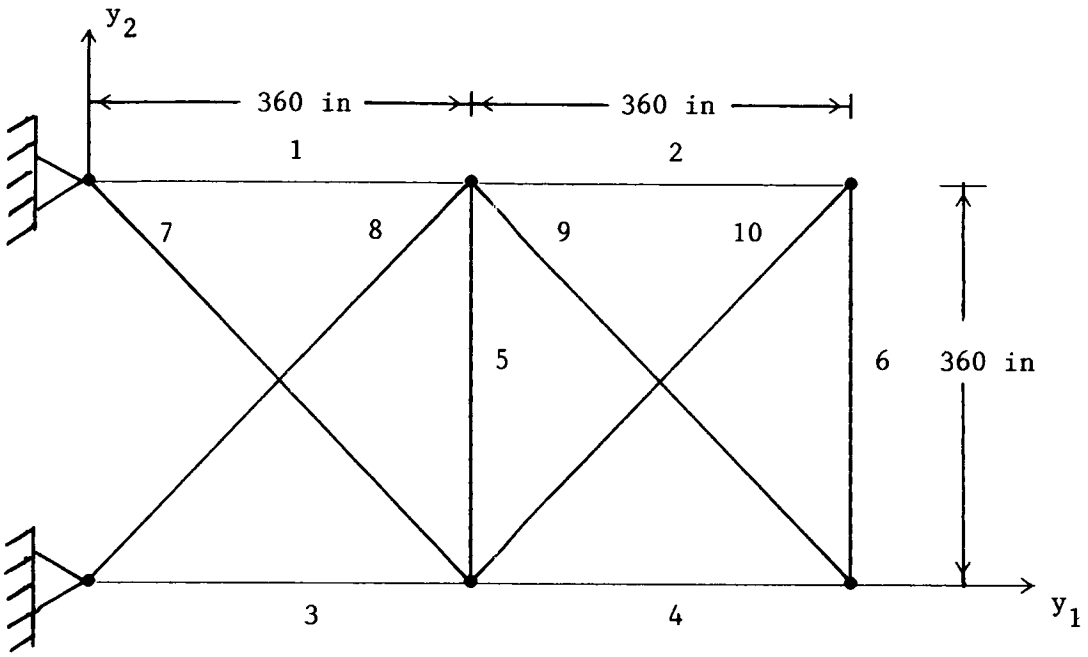
Case No.	$\partial\lambda/\partial x$ Eq. (28)	$\partial\lambda/\partial x$ Exact	Error (%)
A-1	216.7	294.6	26.5
A-2	208.8	269.9	22.5
A-3	202.3	262.7	23.5
B-1	203.8	205.2	0.7
B-2	203.8	222.9	8.6
B-3	196.0	221.0	11.3

TABLE VI. - COMPARISON OF $\partial\lambda_2/\partial x_1$

Case No.	$\partial\lambda/\mu x$ Eq. (28)	$\partial\lambda/\mu x$ Exact	Error (%)
A-1	-812.2	-786.9	3.2
A-2	-747.4	-793.9	5.8
A-3	-764.7	-805.2	9.0
B-1	-1232.3	-1247.8	1.2
B-2	-1387.7	-1320.2	5.1
B-3	-1482.4	-1435.7	3.2

TABLE VII. - COMPARISON OF $\partial\lambda_2/\partial x_2$

Case No.	$\partial\lambda/\partial x$ Eq. (28)	$\partial\lambda/\partial x$ Exact	Error (%)
A-1	3337.9	3652.3	31.35
A-2	3678.3	2917.2	26.1
A-3	3647.5	2876.9	26.8
B-1	3523.5	2928.3	20.3
B-2	3874.9	3038.0	27.5
B-3	4025.3	3043.1	32.3



Modulus of elasticity = 10^4 ksi

Material density = 0.1 lbm/in^3

Design variable x_i = 1,2,3,4

x_2 = 5,6

x_3 = member 7,8,9,10

Figure 1. Ten Member Cantilever Truss